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Original Article

A Comprehensive Study On Fermatean m-Polar Fuzzy Cosets, Fermatean m-Polar Fuzzy Normal Subgroups

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ABSTRACT

This paper presents the concepts of the Fermatean m-Polar Fuzzy order of an element and the Fermatean m-Polar Fuzzy order of a Fermatean m-Polar Fuzzy Subgroup are introduced and investigated. In addition the algebraic properties of Fermatean m-Polar Fuzzy Cosets and Fermatean m-Polar Fuzzy Normal Subgroups (FmPFNSG) are studied, leading to several characterization results and structural theorems. Illustrative examples are included.

Keywords: Fermatean m-Polar Fuzzy Cosets, Fermatean m-Polar Fuzzy Normal Subgroups, Fermatean m-Polar Fuzzy order of an element, Fermatean m-Polar Fuzzy order of a Fermatean m-Polar Fuzzy Subgroup.

INTRODUCTION

Zadeh [10] introduced the concept of fuzzy sets in 1965, establishing a mathematical framework for representing uncertainty through membership functions. This pioneering work laid the foundation for the subsequent development of fuzzy algebra and fuzzy group theory.

Biswas [4] extended Zadeh's fuzzy set theory to algebraic structures by defining fuzzy subgroups and anti-fuzzy subgroups. This work initiated the study of fuzzy groups and inspired further investigations into their structural properties.

Biswas and Choudhury [3] examined the fundamental concepts of fuzzy groups and fuzzy subgroups, presenting several algebraic properties and illustrating how classical group-theoretic notions can be generalized within the fuzzy environment.

Senapathi and Yager [8] introduced Fermatean fuzzy sets as a more flexible extension of intuitionistic and Pythagorean fuzzy sets. Their model allows a wider range of membership and non-membership values while preserving the Fermatean constraint, making it suitable for handling higher levels of uncertainty.

Silambarasan [9] applied the theory of Fermatean fuzzy sets to group theory by defining Fermatean fuzzy subgroups and establishing several of their basic algebraic properties. This work demonstrated the applicability of Fermatean fuzzy concepts in abstract algebra.

Naeem, Riaz, and Karaaslan [5] proposed the notion of Pythagorean m-polar fuzzy sets and investigated their theoretical properties. They also demonstrated the usefulness of these sets in solving decision-making problems involving multiple attributes.

Abdul Razaq [1] studied Pythagorean fuzzy normal subgroups and Pythagorean fuzzy isomorphisms in detail. The paper established important characterizations of normality and homomorphic mappings, thereby enriching the algebraic theory of Pythagorean fuzzy groups.

Bhunia [2] extended the classical Lagrange theorem to Pythagorean fuzzy subgroups by introducing an appropriate notion of subgroup order in the Pythagorean fuzzy setting. The results provide a significant connection between classical finite group theory and Pythagorean fuzzy algebra.

Radharamani and Rajeswari [6] investigated the fundamental properties of Fermatean m-polar fuzzy sets, including their operations and structural characteristics. The study provides a theoretical basis for extending Fermatean fuzzy concepts to more advanced algebraic systems.

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Radharamani and Rajeswari [7] introduced Fermatean m-polar fuzzy subgroups and established several of their algebraic properties. Their work extends the theory of Fermatean m-polar fuzzy sets to group theory and provides a foundation for further research on concepts such as normal subgroups, cosets, and group order in the Fermatean m-polar fuzzy framework.

2.Preliminaries

Definition :2.1[10] Let U be an universal set. A *fuzzy set* F on U is defined by F=$\{\langle u,\varphi(u)\rangle: for\ all\ u \in$ U$\}$,where φ is a function from U $\longrightarrow$ [0,1] and it is called the membership function.

Definition :2.2[3] Let (Ǵ,∗) be a group and U be an universal set. The fuzzy subset F=$\{\langle u,\varphi(u)\rangle\}$ of a group Ǵ is said to be a *fuzzy subgroup* of Ǵ if it satisfies the following conditions

(i) $\varphi(uv) \geq min\{\varphi(u),\varphi(v)\}$

(ii) $\varphi(u^{-1}) \geq \varphi(u)$ for all $u, v \in$ Ǵ

Definition :2.3[5] Let m be any positive integer. A *Pythagorean m-polar fuzzy set* ***(PmFS)*** on U is defined by

$$\mathrm{P} = \left\{\langle u, \left(\varphi_P^{(1)}(u), \Psi_P^{(1)}(u)\right), \left(\varphi_P^{(2)}(u), \Psi_P^{(2)}(u)\right), .., \left(\varphi_P^{(m)}(u), \Psi_P^{(m)}(u)\right)\rangle ; \forall u \in \mathrm{U}\right\},$$

where

$\varphi_P^{(i)}: \mathrm{U} \to [0,1]$ is the membership function $\Psi_P^{(i)}: \mathrm{U} \to [0,1]$ is the non membership function which satisfies the condition $\left(\varphi_P^{(i)}(u)\right)^2 + \left(\Psi_P^{(i)}(u)\right)^2 \leq 1$. The set

$$\mathrm{P} = \left\{\langle u, \left(\varphi_P^{(1)}(u), \Psi_P^{(1)}(u)\right), \left(\varphi_P^{(2)}(u), \Psi_P^{(2)}(u)\right), .., \left(\varphi_P^{(m)}(u), \Psi_P^{(m)}(u)\right)\rangle ; \forall u \in \mathrm{U}\right\}$$

$$\text{is rewrite as } \mathrm{P} = \left\{\langle u, \left(\varphi_P^{(i)}(u), \Psi_P^{(i)}(u)\right)\rangle ; \forall u \in \mathrm{U}, i = 1,2,\ldots,m\right\}$$

Definition :2.4[8] Let U be an universal set. A *Fermatean fuzzy set* F on U is defined by

$F = \{\langle u, (\varphi(u), \Psi(u))\rangle ; \forall u \in \mathrm{U}\}$, where the membership function $\varphi : \mathrm{U} \to [0,1]$ and the non membership function $\Psi : \mathrm{U} \to [0,1]$ which satisfies the condition

$(\varphi(u))^3 + (\Psi(u))^3 \leq 1, \forall u \in \mathrm{U}$.

Definition :2.5[6] Let m be any positive integer and U be an universal set. A *Fermatean m-Polar fuzzy set (FmPFS)* on U is defined as

$$F = \left\{\langle u, \left(\varphi_F^{(1)}(u), \Psi_F^{(1)}(u)\right), \left(\varphi_F^{(2)}(u), \Psi_F^{(2)}(u)\right), .., \left(\varphi_F^{(m)}(u), \Psi_F^{(m)}(u)\right)\rangle ; \forall u \in \mathrm{U}\right\}$$

$$= \left\{\langle u, \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right)\rangle ; \forall u \in \mathrm{U}, i = 1,2,\ldots,m\right\}$$

,where $\varphi_F^{(i)}(u): \mathrm{U} \to [0,1]$ is the membership function $\Psi_F^{(i)}(u): \mathrm{U} \to [0,1]$ is the non membership function which satisfies the condition $\left(\varphi_F^{(i)}(u)\right)^3 + \left(\Psi_F^{(i)}(u)\right)^3 \leq 1$.

Definition :2.6[6] The *intersection* of two Fermatean m-polar fuzzy sets on U is defined by

$$F_1 \cap F_2 = \left\{ \langle u, \left(\min\left(\varphi_{F_1}^{(i)}(u), \varphi_{F_2}^{(i)}(u) \right), \max\left(\Psi_{F_1}^{(i)}(u), \Psi_{F_2}^{(i)}(u) \right) \right) \rangle, i = 1,2,\dots,m \;\&\; \forall u \in \mathrm{U} \right\}$$

Definition :2.7[7] Let (Ǵ, *) be a group and $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,\dots,m$ be a FmPFS. Then F is said to be a Fermatean m-Polar Fuzzy Subgroup (FmPFSG) of G if it satisfies the following conditions

(i) $\left(\varphi_F^{(i)}\right)^3 (u * v) \geq \left(\varphi_F^{(i)}\right)^3 (u) \wedge \left(\varphi_F^{(i)}\right)^3 (v)$

(ii) $\left(\varphi_F^{(i)}\right)^3 (u^{-1}) \geq \left(\varphi_F^{(i)}\right)^3 (u)$

(iii) $\left(\Psi_F^{(i)}\right)^3 (u * v) \leq \left(\Psi_F^{(i)}\right)^3 (u) \vee \left(\Psi_F^{(i)}\right)^3 (v)$

(iv) $\left(\Psi_F^{(i)}\right)^3 (u^{-1}) \leq \left(\Psi_F^{(i)}\right)^3 (u)$, $i = 1,2,\dots,m$ and for all $u \in$ Ǵ

Here $\left(\varphi_F^{(i)}\right)^3 (u) = \left(\varphi_F^{(i)}(u)\right)^3$ and $\left(\Psi_F^{(i)}\right)^3 (u) = \left(\Psi_F^{(i)}(u)\right)^3$

Definition :2.8[1] Let $P = \{\langle u, (\varphi_P(u), \Psi_P(u))\rangle : u \in$ Ǵ$\}$ be a Pythagorean Fuzzy Subgroup of Ǵ(PFSG). For any element $a \in$ Ǵ the Pythagorean fuzzy left coset of P associated with a is represented by $aP = \{\langle u, (\varphi_{aP}(u), \Psi_{aP}(u))\rangle : u \in$ Ǵ$\}$, where $(\varphi_{aP}(u))^2 = (\varphi_P(a^{-1}u))^2$ and $(\Psi_{aP}(u))^2 = (\Psi_P(a^{-1}u))^2$. Likewise, the Pythagorean fuzzy right coset of P corresponding to the element a and is denoted by $Pa = \{\langle u, (\varphi_{Pa}(u), \Psi_{Pa}(u))\rangle : u \in$ Ǵ$\}$, where $(\Psi_{Pa}(u))^2 = (\Psi_P(ua^{-1}))^2$ and $(\Psi_{Pa}(u))^2 = (\Psi_P(ua^{-1}))^2$.

Definition :2.9[1] Let PFSG $P = \{\langle u, (\varphi_P(u), \Psi_P(u))\rangle : u \in$ Ǵ$\}$ of Ǵ.Then P is said to be a Pythagorean fuzzy normal subgroup of Ǵ if $aP = Pa$ for all $a \in$ Ǵ.

Theorem:2.10[2] Let PFSG $P = \{\langle u, (\varphi_P(u), \Psi_P(u))\rangle : u \in$ Ǵ$\}$ of Ǵ and $v \in$ Ǵ.Then the subset $\Gamma(v) = \{u \in$ Ǵ$/ \varphi_P{}^2(v) \geq \varphi_P{}^2(u)$ and $\Psi_P{}^2(v) \leq \Psi_P{}^2(u)\}$ is a Pythagorean fuzzy subgroup of Ǵ.

Definition:2.10[2] Let PFSG $P = \{\langle u, (\varphi_P(u), \Psi_P(u))\rangle : u \in$ Ǵ$\}$ of Ǵ and $v \in$ Ǵ.Then the subgroup $\Gamma(v) = \{u \in$ Ǵ$/ \varphi_P{}^2(v) \geq \varphi_P{}^2(u)$ and $\Psi_P{}^2(v) \leq \Psi_P{}^2(u)\}$ is called the Pythagorean fuzzy semi-level subgroup of Ǵ corresponding to v.

Definition:2.11[2] Suppose $P = (\varphi_P(u), \Psi_P(u))$ is a PFSG of Ǵ and $a \in$ Ǵ.Then the Pythagorean fuzzy order (PFO) of a in P is represented by PFO$(a)_P$, is taken to be the order of the corresponding semi-level subgroup. That is PFO$(a)_P = o(\Gamma(a))$ for all $a \in$ Ǵ.

3.Fermatean *m*-Polar Fuzzy Cosets

Definition :3.1

Let (Ǵ,$*$) be a group and $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG. Let $a \in$ Ǵ, then the *Fermatean m-Polar fuzzy left coset* of F is defined by

$aF = \left\{\langle u, \left(\varphi_{aF}^{(i)}(u), \Psi_{aF}^{(i)}(u)\right)\rangle, , i = 1,2,\dots,m\right\}$, for all $u \in$ Ǵ,

where $\left(\varphi_{aF}^{(i)}(u)\right)^3 = \left(\varphi_F^{(i)}(a^{-1}u)\right)^3$ and $\left(\Psi_{aF}^{(i)}(u)\right)^3 = \left(\Psi_F^{(i)}(a^{-1}u),\right)^3$ for all $u \in \breve{G}$ and also the F*ermatean m-Polar fuzzy right coset* of F is defined by

$$Fa = \{\left(\varphi_{Fa}^{(i)}(u), \Psi_{Fa}^{(i)}(u)\right), i = 1,23,\dots,m,\}, \text{ for all } u \in \text{Ǵ},$$

where $\varphi_{Fa}^{(i)}(u) = \varphi_F^{(i)}(ua^{-1})$ and $\Psi_{Fa}^{(i)}(u) = \Psi_F^{(i)}(ua^{-1})$, for all $u \in$ Ǵ.

Definition :3.2

Let (Ǵ,$*$) be a group and $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG of (Ǵ,$*$).

Then F is said to be *Fermatean m-Polar fuzzy normal subgroup* Ǵ (FmPFNSG) if $aF = Fa$ for all $u \in Ǵ$.

Theorem:3.3

Assume $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ be a FmPFSG on $(Ǵ,*)$ and $u, v \in Ǵ$.Then

$$'\Omega(u) = \left\{v \in Ǵ \ / \ \left(\varphi_F^{(i)}\right)^3(v) \geq \left(\varphi_F^{(i)}\right)^3(u) \ and \ \left(\Psi_F^{(i)}\right)^3(v) \leq \left(\Psi_F^{(i)}\right)^3(u), i = 1,2,3, \dots, m\right\}$$

is a subgroup of $(Ǵ,*)$.

Proof:

Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ be a FmPFSG on $(Ǵ,*)$ and

$$'\Omega(u) = \left\{v \in Ǵ \ / \ \left(\varphi_F^{(i)}\right)^3(v) \geq \left(\varphi_F^{(i)}\right)^3(u) \ and \ \left(\Psi_F^{(i)}\right)^3(v) \leq \left(\Psi_F^{(i)}\right)^3(u), i = 1,2,3, \dots, m\right\}$$

where $u, v \in Ǵ$.

Since $\left(\varphi_F^{(i)}\right)^3(e) \geq \left(\varphi_F^{(i)}\right)^3(u) \ and \ \left(\Psi_F^{(i)}\right)^3(e) \leq \left(\Psi_F^{(i)}\right)^3(u)$,where $'e'$ is the identity element of $(Ǵ,*)$, then $e \in {'\Omega}(u)$ and also $'\Omega(u) \subseteq Ǵ$. Let $u, v \in Ǵ$.

$\left(\varphi_F^{(i)}\right)^3(u * v^{-1}) \geq \left(\varphi_F^{(i)}\right)^3(u) \wedge \left(\varphi_F^{(i)}\right)^3(v) \geq \left(\varphi_F^{(i)}\right)^3(u)$ and

$\left(\Psi_F^{(i)}\right)^3(u * v^{-1}) \leq \left(\Psi_F^{(i)}\right)^3(u) \vee \left(\Psi_F^{(i)}\right)^3(v) \leq \left(\Psi_F^{(i)}\right)^3(u)$

This implies, $u * v^{-1} \in {'\Omega}(u)$

Therefore, $'\Omega(u)$ is a subgroup of $(Ǵ,*)$.

Definition :3.4

Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ be a FmPFSG on $(Ǵ,*)$.Then $'\Omega(u)$ is called the *Fermatean m-polar semi level subgroup* of $(Ǵ,*)$ corresponding to u.

Definition:3.5

Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ be a FmPFSG on $(Ǵ,*)$ and $a \in Ǵ$.Then the Fermatean m-polar fuzzy order of a is defined by the number of elements in the Fermatean m-polar semi level subgroup of $(Ǵ,*)$ corresponding to a and it is denoted by FmPFO(a).

i.e.,$FmPFO(a) = O({'\Omega}(a))$, for all $a \in Ǵ$.

Example:3.6 Let us consider the set $Ǵ = Z_4 = \{0,1,2,3\}$,then $(Ǵ, \oplus_4)$ is a group, where '$\oplus_4'$ is addition modulo 4. Now we define a F3PFS $F = \left\{\langle u, \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right)\rangle, i = 1,2,3\right\}$

on Ǵ by

$$F = \begin{Bmatrix} \langle 0, (1,0), (0.8,0.3)(0.7,0.4)\rangle, \langle 1, (0.5,0.4), (0.4,0.5)(0.2,0.6)\rangle, \\ \langle 2, (1,0), (0.8,0.3)(0.7,0.4)\rangle, \langle 3, (0.5,0.4), (0.4,0.5)(0.2,0.6)\rangle \end{Bmatrix}.$$

Clearly, F is a Fermatean m-Polar Fuzzy Subgroup on $(Ǵ, \oplus_4)$.Then $FmPFO$ of the elements of Z_4 in F is presented by

$$FmPFO(0) = O({'\Omega}(0)) = 2,$$
$$FmPFO(1) = O({'\Omega}(1)) = 4$$
$$FmPFO(2) = O({'\Omega}(2)) = 2,$$
$$FmPFO(3) = O({'\Omega}(3)) = 4$$

From above example, we see that $FmPFO(0) \neq O(0)$.

Remark.3.7 The $FmPFO$ of an element in $FmPFSG$ on a group Ǵ may not always be same to the element's order in the group.

Definition:3.8 Let F be a FmPFSG on a group $(Ǵ,*)$. The index of F is the number of distinct left coset of F in Ǵ and is denoted by $[Ǵ : F]$

Example:3.9

Let us take the group $(\acute{G},*)$, where $\acute{G}=\{1,-1,i,-i\}$ and '*' is a multiplication. Now we define a F2PFS $F=\left\{\langle u,\left(\varphi_F^{(i)}(u),\Psi_F^{(i)}(u)\right)\rangle,,i=1,2\right\}$ of $(\acute{G},*)$ by

$\varphi_F^{(1)}(1)=0.6$, $\Psi_F^{(1)}(1)=0.1$; $\varphi_F^{(2)}(1)=0.7$, $\Psi_F^{(2)}(1)=0.3$

$\varphi_F^{(1)}(-1)=0.5$, $\Psi_F^{(1)}(-1)=0.4$; $\varphi_F^{(2)}(-1)=0.7$, $\Psi_F^{(2)}(-1)=0.6$

$\varphi_F^{(1)}(i)=0.4$, $\Psi_F^{(1)}(i)=0.5$; $\varphi_F^{(2)}(i)=0.6$, $\Psi_F^{(2)}(i)=0.7$

$\varphi_F^{(1)}(-i)=0.4$, $\Psi_F^{(1)}(-i)=0.5$; $\varphi_F^{(2)}(-i)=0.6$, $\Psi_F^{(2)}(-i)=0.7$

Then F is a FmPFSG of $\acute{G}$.

Next, we find the Fermatean m-Polar fuzzy left cosets F with respect to all $u\in\acute{G}$.

(i) The *Fermatean m-Polar fuzzy left cosets* of F with respect to $1\in\acute{G}$.

$$1F=\left\{\begin{array}{c}\langle 1,\left(\varphi_F^{(1)}(1^{-1}*1),\Psi_F^{(1)}(1^{-1}*1)\right),\left(\varphi_F^{(2)}(1^{-1}*1),\Psi_F^{(2)}(1^{-1}*1)\right)\rangle,\\ \langle -1,\left(\varphi_F^{(1)}(1^{-1}*-1),\Psi_F^{(1)}(1^{-1}*-1)\right),\left(\varphi_F^{(2)}(1^{-1}*-1),\Psi_F^{(2)}(1^{-1}*-1)\right)\rangle,\\ \langle i,\left(\varphi_F^{(1)}(1^{-1}*i),\Psi_F^{(1)}(1^{-1}*i)\right),\left(\varphi_F^{(2)}(1^{-1}*i),\Psi_F^{(2)}(1^{-1}*i)\right)\rangle,\\ \langle -i,\left(\varphi_F^{(1)}(1^{-1}*-i),\Psi_F^{(1)}(1^{-1}*-i)\right),\left(\varphi_F^{(2)}(1^{-1}*-i),\Psi_F^{(2)}(1^{-1}*-i)\right)\rangle\end{array}\right\}$$

$$=\left\{\begin{array}{c}\langle 1,\left(\varphi_F^{(1)}(1),\Psi_F^{(1)}(1)\right),\left(\varphi_F^{(2)}(1),\Psi_F^{(2)}(1)\right)\rangle,\\ \langle -1,\left(\varphi_F^{(1)}(-1),\Psi_F^{(1)}(-1)\right),\left(\varphi_F^{(2)}(-1),\Psi_F^{(2)}(-1)\right)\rangle,\\ \langle i,\left(\varphi_F^{(1)}(i),\Psi_F^{(1)}(i)\right),\left(\varphi_F^{(2)}(i),\Psi_F^{(2)}(i)\right)\rangle,\\ \langle -i,\left(\varphi_F^{(1)}(-i),\Psi_F^{(1)}(-i)\right),\left(\varphi_F^{(2)}(-i),\Psi_F^{(2)}(-i)\right)\rangle\end{array}\right\}$$

$$=\left\{\begin{array}{c}\langle 1,(0.6,0.1),(0.7,0.3)\rangle,\langle -1,(0.5,0.4),(0.7,0.6)\rangle,\\ \langle i,(0.4,0.5),(0.6,0.7)\rangle,\langle -i,(0.4,0.5),(0.6,0.7)\rangle\end{array}\right\}$$

(ii) The *Fermatean m-Polar fuzzy left cosets* of F with respect to $-1\in\acute{G}$.

$$-1F=\left\{\begin{array}{c}\langle 1,\left(\varphi_F^{(1)}(-1^{-1}*1),\Psi_F^{(1)}(-1^{-1}*1)\right),\left(\varphi_F^{(2)}(-1^{-1}*1),\Psi_F^{(2)}(-1^{-1}*1)\right)\rangle,\\ \langle -1,\left(\varphi_F^{(1)}(-1^{-1}*-1),\Psi_F^{(1)}(-1^{-1}*-1)\right),\left(\varphi_F^{(2)}(-1^{-1}*-1),\Psi_F^{(2)}(-1^{-1}*-1)\right)\rangle,\\ \langle i,\left(\varphi_F^{(1)}(-1^{-1}*i),\Psi_F^{(1)}(-1^{-1}*i)\right),\left(\varphi_F^{(2)}(-1^{-1}*i),\Psi_F^{(2)}(-1^{-1}*i)\right)\rangle,\\ \langle -i,\left(\varphi_F^{(1)}(-1^{-1}*-i),\Psi_F^{(1)}(-1^{-1}*-i)\right),\left(\varphi_F^{(2)}(-1^{-1}*-i),\Psi_F^{(2)}(-1^{-1}*-i)\right)\rangle\end{array}\right\}$$

$$= \left\{\begin{array}{c}\langle 1, \left(\varphi_F^{(1)}(-1), \Psi_F^{(1)}(-1)\right), \left(\varphi_F^{(2)}(-1), \Psi_F^{(2)}(-1)\right)\rangle, \\ \langle -1, \left(\varphi_F^{(1)}(1), \Psi_F^{(1)}(1)\right), \left(\varphi_F^{(2)}(1), \Psi_F^{(2)}(1)\right)\rangle, \\ \langle i, \left(\varphi_F^{(1)}(-i), \Psi_F^{(1)}(-i)\right), \left(\varphi_F^{(2)}(-i), \Psi_F^{(2)}(-i)\right)\rangle, \\ \langle -i, \left(\varphi_F^{(1)}(i), \Psi_F^{(1)}(i)\right), \left(\varphi_F^{(2)}(i), \Psi_F^{(2)}(i)\right)\rangle\end{array}\right\}$$

$$= \left\{\begin{array}{c}\langle 1, (0.5,0.4), (0.7,0.6)\rangle, \langle -1, (0.6,0.1), (0.7,0.3)\rangle, \\ \langle i, (0.4,0.5), (0.6,0.7)\rangle, \langle -i, (0.4,0.5), (0.6,0.7)\rangle\end{array}\right\}$$

(iii) The *Fermatean m-Polar fuzzy left cosets* of F with respect to $i \in \acute{G}$

$$iF = \left\{\begin{array}{c}\langle 1, \left(\varphi_F^{(1)}(i^{-1} * 1), \Psi_F^{(1)}(i^{-1} * 1)\right), \left(\varphi_F^{(2)}(i^{-1} * 1), \Psi_F^{(2)}(i^{-1} * 1)\right)\rangle, \\ \langle -1, \left(\varphi_F^{(1)}(i^{-1} * -1), \Psi_F^{(1)}(i^{-1} * -1)\right), \left(\varphi_F^{(2)}(i^{-1} * -1), \Psi_F^{(2)}(i^{-1} * -1)\right)\rangle, \\ \langle i, \left(\varphi_F^{(1)}(i^{-1} * i), \Psi_F^{(1)}(i^{-1} * i)\right), \left(\varphi_F^{(2)}(i^{-1} * i), \Psi_F^{(2)}(i^{-1} * i)\right)\rangle, \\ \langle -i, \left(\varphi_F^{(1)}(i^{-1} * -i), \Psi_F^{(1)}(i^{-1} * -i)\right), \left(\varphi_F^{(2)}(i^{-1} * -i), \Psi_F^{(2)}(i^{-1} * -i)\right)\rangle\end{array}\right\}$$

$$= \left\{\begin{array}{c}\langle 1, \left(\varphi_F^{(1)}(-i), \Psi_F^{(1)}(-i)\right), \left(\varphi_F^{(2)}(-i), \Psi_F^{(2)}(-i)\right)\rangle, \\ \langle -1, \left(\varphi_F^{(1)}(i), \Psi_F^{(1)}(i)\right), \left(\varphi_F^{(2)}(i), \Psi_F^{(2)}(i)\right)\rangle, \\ \langle i, \left(\varphi_F^{(1)}(1), \Psi_F^{(1)}(1)\right), \left(\varphi_F^{(2)}(1), \Psi_F^{(2)}(1)\right)\rangle, \\ \langle -i, \left(\varphi_F^{(1)}(-1), \Psi_F^{(1)}(-1)\right), \left(\varphi_F^{(2)}(-1), \Psi_F^{(2)}(-1)\right)\rangle\end{array}\right\}$$

$$= \left\{\begin{array}{c}\langle 1, (0.4,0.5), (0.6,0.7)\rangle, \langle -1, (0.4,0.5), (0.6,0.7)\rangle, \\ \langle i, (0.6,0.1), (0.7,0.3)\rangle, \langle -i, (0.5,0.4), (0.7,0.6)\rangle\end{array}\right\}$$

(iv) The *Fermatean m-Polar fuzzy left cosets* of F with respect to $-i \in \acute{G}$

$$-iF$$

$$= \left\{\begin{array}{c}\langle 1, \left(\varphi_F^{(1)}(-i^{-1} * 1), \Psi_F^{(1)}(-i^{-1} * 1)\right), \left(\varphi_F^{(2)}(-i^{-1} * 1), \Psi_F^{(2)}(-i^{-1} * 1)\right)\rangle, \\ \langle -1, \left(\varphi_F^{(1)}(-i^{-1} * -1), \Psi_F^{(1)}(-i^{-1} * -1)\right), \left(\varphi_F^{(2)}(-i^{-1} * -1), \Psi_F^{(2)}(-i^{-1} * -1)\right)\rangle, \\ \langle i, \left(\varphi_F^{(1)}(-i^{-1} * i), \Psi_F^{(1)}(-i^{-1} * i)\right), \left(\varphi_F^{(2)}(-i^{-1} * i), \Psi_F^{(2)}(-i^{-1} * i)\right)\rangle, \\ \langle -i, \left(\varphi_F^{(1)}(-i^{-1} * -i), \Psi_F^{(1)}(-i^{-1} * -i)\right), \left(\varphi_F^{(2)}(-i^{-1} * -i), \Psi_F^{(2)}(-i^{-1} * -i)\right)\rangle\end{array}\right\}$$

$$= \left\{\begin{array}{c}\langle 1, \left(\varphi_F^{(1)}(i), \Psi_F^{(1)}(i)\right), \left(\varphi_F^{(2)}(i), \Psi_F^{(2)}(i)\right)\rangle, \\ \langle -1, \left(\varphi_F^{(1)}(-i), \Psi_F^{(1)}(-i)\right), \left(\varphi_F^{(2)}(-i), \Psi_F^{(2)}(-i)\right)\rangle, \\ \langle i, \left(\varphi_F^{(1)}(-1), \Psi_F^{(1)}(-1)\right), \left(\varphi_F^{(2)}(-1), \Psi_F^{(2)}(-1)\right)\rangle, \\ \langle -i, \left(\varphi_F^{(1)}(1), \Psi_F^{(1)}(1)\right), \left(\varphi_F^{(2)}(1), \Psi_F^{(2)}(1)\right)\rangle\end{array}\right\}$$

$$= \begin{Bmatrix} \langle 1,(0.4,0.5),(0.6,0.7)\rangle, \langle -1,(0.4,0.5),(0.6,0.7)\rangle, \\ \langle i,(0.5,0.4),(0.7,0.6)\rangle, \langle -i,(0.6,0.1),(0.7,0.3)\rangle \end{Bmatrix}$$

Now, we find the *Fermatean m-Polar fuzzy right cosets* F with respect to all $u \in \acute{G}$.

(i) The *Fermatean m-Polar fuzzy right cosets* of F with respect to $1 \in \acute{G}$.

$$F1 = \begin{Bmatrix} \langle 1,\left(\varphi_F^{(1)}(1*1^{-1}),\Psi_F^{(1)}(1*1^{-1})\right),\left(\varphi_F^{(2)}(1*1^{-1}),\Psi_F^{(2)}(1*1^{-1})\right)\rangle, \\ \langle -1,\left(\varphi_F^{(1)}(-1*1^{-1}),\Psi_F^{(1)}(-1*1^{-1})\right),\left(\varphi_F^{(2)}(-1*1^{-1}),\Psi_F^{(2)}(-1*1^{-1})\right)\rangle, \\ \langle i,\left(\varphi_F^{(1)}(i*1^{-1}),\Psi_F^{(1)}(i*1^{-1})\right),\left(\varphi_F^{(2)}(i*1^{-1}),\Psi_F^{(2)}(i*1^{-1})\right)\rangle, \\ \langle -i,\left(\varphi_F^{(1)}(-i*1^{-1}),\Psi_F^{(1)}(-i*1^{-1})\right),\left(\varphi_F^{(2)}(-i*1^{-1}),\Psi_F^{(2)}(-i*1^{-1})\right)\rangle \end{Bmatrix}$$

$$= \begin{Bmatrix} \langle 1,\left(\varphi_F^{(1)}(1),\Psi_F^{(1)}(1)\right),\left(\varphi_F^{(2)}(1),\Psi_F^{(2)}(1)\right)\rangle, \\ \langle -1,\left(\varphi_F^{(1)}(-1),\Psi_F^{(1)}(-1)\right),\left(\varphi_F^{(2)}(-1),\Psi_F^{(2)}(-1)\right)\rangle, \\ \langle i,\left(\varphi_F^{(1)}(i),\Psi_F^{(1)}(i)\right),\left(\varphi_F^{(2)}(i),\Psi_F^{(2)}(i)\right)\rangle, \\ \langle -i,\left(\varphi_F^{(1)}(-i),\Psi_F^{(1)}(-i)\right),\left(\varphi_F^{(2)}(-i),\Psi_F^{(2)}(-i)\right)\rangle \end{Bmatrix}$$

$$= \begin{Bmatrix} \langle 1,(0.6,0.1),(0.7,0.3)\rangle, \langle -1,(0.5,0.4),(0.7,0.6)\rangle, \\ \langle i,(0.4,0.5),(0.6,0.7)\rangle, \langle -i,(0.4,0.5),(0.6,0.7)\rangle \end{Bmatrix}$$

(ii) The *Fermatean m-Polar fuzzy right cosets* of F with respect to $-1 \in \acute{G}$.

$$F(-1) = \begin{Bmatrix} \langle 1,\left(\varphi_F^{(1)}(1*-1^{-1}),\Psi_F^{(1)}(1*-1^{-1})\right),\left(\varphi_F^{(2)}(1*-1^{-1}),\Psi_F^{(2)}(1*-1^{-1})\right)\rangle, \\ \langle -1,\left(\varphi_F^{(1)}(-1*-1^{-1}),\Psi_F^{(1)}(-1*-1^{-1})\right),\left(\varphi_F^{(2)}(-1*-1^{-1}),\Psi_F^{(2)}(-1*-1^{-1})\right)\rangle, \\ \langle i,\left(\varphi_F^{(1)}(i*-1^{-1}),\Psi_F^{(1)}(i*-1^{-1})\right),\left(\varphi_F^{(2)}(i*-1^{-1}),\Psi_F^{(2)}(i*-1^{-1})\right)\rangle, \\ \langle -i,\left(\varphi_F^{(1)}(-i*-1^{-1}),\Psi_F^{(1)}(-i*-1^{-1})\right),\left(\varphi_F^{(2)}(-i*-1^{-1}),\Psi_F^{(2)}(-i*-1^{-1})\right)\rangle \end{Bmatrix}$$

$$= \begin{Bmatrix} \langle 1,\left(\varphi_F^{(1)}(-1),\Psi_F^{(1)}(-1)\right),\left(\varphi_F^{(2)}(-1),\Psi_F^{(2)}(-1)\right)\rangle, \\ \langle -1,\left(\varphi_F^{(1)}(1),\Psi_F^{(1)}(1)\right),\left(\varphi_F^{(2)}(1),\Psi_F^{(2)}(1)\right)\rangle, \\ \langle i,\left(\varphi_F^{(1)}(-i),\Psi_F^{(1)}(-i)\right),\left(\varphi_F^{(2)}(-i),\Psi_F^{(2)}(-i)\right)\rangle, \\ \langle -i,\left(\varphi_F^{(1)}(i),\Psi_F^{(1)}(i)\right),\left(\varphi_F^{(2)}(i),\Psi_F^{(2)}(i)\right)\rangle \end{Bmatrix}$$

$$= \begin{Bmatrix} \langle 1,(0.5,0.4),(0.7,0.6)\rangle, \langle -1,(0.6,0.1),(0.7,0.3)\rangle, \\ \langle i,(0.4,0.5),(0.6,0.7)\rangle, \langle -i,(0.4,0.5),(0.6,0.7)\rangle \end{Bmatrix}$$

(iii) The *Fermatean m-Polar fuzzy right cosets* of F with respect to $i \in \acute{G}$

$$Fi=\begin{Bmatrix}\langle 1,\left(\varphi_F^{(1)}(1*i^{-1}),\Psi_F^{(1)}(1*i^{-1})\right),\left(\varphi_F^{(2)}(1*i^{-1}),\Psi_F^{(2)}(1*i^{-1})\right)\rangle,\\ \langle -1,\left(\varphi_F^{(1)}(-1*i^{-1}),\Psi_F^{(1)}(-1*i^{-1})\right),\left(\varphi_F^{(2)}(-1*i^{-1}),\Psi_F^{(2)}(-1*i^{-1})\right)\rangle,\\ \langle i,\left(\varphi_F^{(1)}(i*i^{-1}),\Psi_F^{(1)}(i*i^{-1})\right),\left(\varphi_F^{(2)}(i*i^{-1}),\Psi_F^{(2)}(i*i^{-1})\right)\rangle,\\ \langle -i,\left(\varphi_F^{(1)}(-i*i^{-1}),\Psi_F^{(1)}(-i*i^{-1})\right),\left(\varphi_F^{(2)}(-i*i^{-1}),\Psi_F^{(2)}(-i*i^{-1})\right)\rangle\end{Bmatrix}$$

$$=\begin{Bmatrix}\langle 1,\left(\varphi_F^{(1)}(-i),\Psi_F^{(1)}(-i)\right),\left(\varphi_F^{(2)}(-i),\Psi_F^{(2)}(-i)\right)\rangle,\\ \langle -1,\left(\varphi_F^{(1)}(i),\Psi_F^{(1)}(i)\right),\left(\varphi_F^{(2)}(i),\Psi_F^{(2)}(i)\right)\rangle,\\ \langle i,\left(\varphi_F^{(1)}(1),\Psi_F^{(1)}(1)\right),\left(\varphi_F^{(2)}(1),\Psi_F^{(2)}(1)\right)\rangle,\\ \langle -i,\left(\varphi_F^{(1)}(-1),\Psi_F^{(1)}(-1)\right),\left(\varphi_F^{(2)}(-1),\Psi_F^{(2)}(-1)\right)\rangle\end{Bmatrix}$$

$$=\begin{Bmatrix}\langle 1,(0.4,0.5),(0.6,0.7)\rangle,\langle -1,(0.4,0.5),(0.6,0.7)\rangle,\\ \langle i,(0.6,0.1),(0.7,0.3)\rangle,\langle -i,(0.5,0.4),(0.7,0.6)\rangle\end{Bmatrix}$$

(iv)The *Fermatean m-Polar fuzzy right cosets* of F with respect to $-i\in Ǵ$

$$F(-i)$$

$$=\begin{Bmatrix}\langle 1,\left(\varphi_F^{(1)}(1*-i^{-1}),\Psi_F^{(1)}(1*-i^{-1})\right),\left(\varphi_F^{(2)}(1*-i^{-1}),\Psi_F^{(2)}(1*-i^{-1})\right)\rangle,\\ \langle -1,\left(\varphi_F^{(1)}(-1*-i^{-1}),\Psi_F^{(1)}(-1*-i^{-1})\right),\left(\varphi_F^{(2)}(-1*-i^{-1}),\Psi_F^{(2)}(-1*-i^{-1})\right)\rangle,\\ \langle i,\left(\varphi_F^{(1)}(i*-i^{-1}),\Psi_F^{(1)}(i*-i^{-1})\right),\left(\varphi_F^{(2)}(i*-i^{-1}),\Psi_F^{(2)}(i*-i^{-1})\right)\rangle,\\ \langle -i,\left(\varphi_F^{(1)}(-i*-i^{-1}),\Psi_F^{(1)}(-i*-i^{-1})\right),\left(\varphi_F^{(2)}(-i*-i^{-1}),\Psi_F^{(2)}(-i*-i^{-1})\right)\rangle\end{Bmatrix}$$

$$=\begin{Bmatrix}\langle 1,\left(\varphi_F^{(1)}(i),\Psi_F^{(1)}(i)\right),\left(\varphi_F^{(2)}(i),\Psi_F^{(2)}(i)\right)\rangle,\\ \langle -1,\left(\varphi_F^{(1)}(-i),\Psi_F^{(1)}(-i)\right),\left(\varphi_F^{(2)}(-i),\Psi_F^{(2)}(-i)\right)\rangle,\\ \langle i,\left(\varphi_F^{(1)}(-1),\Psi_F^{(1)}(-1)\right),\left(\varphi_F^{(2)}(-1),\Psi_F^{(2)}(-1)\right)\rangle,\\ \langle -i,\left(\varphi_F^{(1)}(1),\Psi_F^{(1)}(1)\right),\left(\varphi_F^{(2)}(1),\Psi_F^{(2)}(1)\right)\rangle\end{Bmatrix}$$

$$=\begin{Bmatrix}\langle 1,(0.4,0.5),(0.6,0.7)\rangle,\langle -1,(0.4,0.5),(0.6,0.7)\rangle,\\ \langle i,(0.5,0.4),(0.7,0.6)\rangle,\langle -i,(0.6,0.1),(0.7,0.3)\rangle\end{Bmatrix}$$

This implies F is Fermatean m-Polar fuzzy normal subgroup Ǵ.

In this example, there are four distinct left cosets of F with respect to Ǵ and $aF = Fa$ for all $a \in Ǵ$.

Therefore, index of F= 4.

Proposition:3.10

Let $F=\left(\varphi_F^{(i)}(u),\Psi_F^{(i)}(u)\right), i=1,2,3,\dots,m$ be a FmPFSG on $(Ǵ,*)$.Then F is FmPFNSG on $(Ǵ,*)$ iff $\left(\varphi_F^{(i)}\right)^3(u*v)=\left(\varphi_F^{(i)}\right)^3(v*u)$ and $\left(\Psi_F^{(i)}\right)^3(u*v)=\left(\Psi_F^{(i)}\right)^3(v*u)$,

$i=1,2,3,\dots,m$ and for all $u,v\in Ǵ$.

Proof. Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ be a FmPFNSG on $(\acute{G},*)$.Then $aF = Fa$ for all $a \in \acute{G}$.This implies $\left(\varphi_{aF}^{(i)}\right)^3(u) = \left(\varphi_{Fa}^{(i)}\right)^3(u)$ and $\left(\Psi_{aF}^{(i)}\right)^3(u) = \left(\Psi_{Fa}^{(i)}\right)^3(u)$.This implies $\left(\varphi_F^{(i)}(a^{-1} * u)\right)^3 = \left(\varphi_F^{(i)}(u * a^{-1})\right)^3$ and $\left(\Psi_F^{(i)}(a^{-1} * u)\right)^3 = \left(\Psi_F^{(i)}(u * a^{-1})\right)^3$, $i = 1,2,3, \dots, m$, for all $a, u \in \acute{G}$. Now $\left(\varphi_F^{(i)}\right)^3(u * v) = \left(\varphi_F^{(i)}\right)^3(u * (v^{-1})^{-1}) = \left(\varphi_F^{(i)}\right)^3((v^{-1})^{-1} * u) = \left(\varphi_F^{(i)}\right)^3(v * u)$.

Conversely, let us assume that $\left(\varphi_F^{(i)}\right)^3(u * v) = \left(\varphi_F^{(i)}\right)^3(v * u)$ and $\left(\Psi_F^{(i)}\right)^3(u * v) = \left(\Psi_F^{(i)}\right)^3(v * u), i = 1,2,3, \dots, m$ and for all $u, v \in \acute{G}$.Let $a \in \acute{G}$. Since $\acute{G}$ is a group, then $a^{-1} \in \acute{G}$ and take $a^{-1} = k$. Now $\left(\varphi_F^{(i)}\right)^3(k * u) = \left(\varphi_F^{(i)}\right)^3(u * k)$ and $\left(\Psi_F^{(i)}\right)^3(k * u) = \left(\Psi_F^{(i)}\right)^3(u * k)$. This implies $\left(\varphi_F^{(i)}\right)^3(a^{-1} * u) = \left(\varphi_F^{(i)}\right)^3(u * a^{-1})$ and $\left(\Psi_F^{(i)}\right)^3(a^{-1} * u) = \left(\Psi_F^{(i)}\right)^3(u * a^{-1})$. This implies $\left(\varphi_{aF}^{(i)}\right)^3(u) = \left(\varphi_{Fa}^{(i)}\right)^3(u)$ and $\left(\Psi_{aF}^{(i)}\right)^3(u) = \left(\Psi_{Fa}^{(i)}\right)^3(u)$. Therefore, $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ is a FmPFNSG on $(\acute{G},*)$

Proposition:3.11 Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ be a FmPFSG on $(\acute{G},*)$.Then F is FmPFNSG on $(\acute{G},*)$ iff $\left(\varphi_F^{(i)}\right)^3(g * u * g^{-1}) = \left(\varphi_F^{(i)}\right)^3(u)$ and $\left(\Psi_F^{(i)}\right)^3(g * u * g^{-1}) = \left(\Psi_F^{(i)}\right)^3(u)$,

$i = 1,2,3, \dots, m$ and for all $u, g \in \acute{G}$.

Proof. Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \dots, m$ be a FmPFNSG on $(\acute{G},*)$.Then $\left(\varphi_F^{(i)}\right)^3(u * v) = \left(\varphi_F^{(i)}\right)^3(v * u)$ and $\left(\Psi_F^{(i)}\right)^3(u * v) = \left(\Psi_F^{(i)}\right)^3(v * u), i = 1,2,3, \dots, m$ and for all $u, v \in \acute{G}$. Now

$$\left(\varphi_F^{(i)}\right)^3(g * u * g^{-1}) = \left(\varphi_F^{(i)}\right)^3((g * u) * g^{-1})$$

$$= \left(\varphi_F^{(i)}\right)^3(g^{-1} * g * u)$$

$$= \left(\varphi_F^{(i)}\right)^3(u)$$

Similarly, we can prove $\left(\Psi_F^{(i)}\right)^3(g * u * g^{-1}) = \left(\Psi_F^{(i)}\right)^3(u)$, $i = 1,2,3, \dots, m$ and for all $u, g \in \acute{G}$.

Conversely, let us assume that $\left(\varphi_F^{(i)}\right)^3(g * u * g^{-1}) = \left(\varphi_F^{(i)}\right)^3(u)$ and

$\left(\Psi_F^{(i)}\right)^3(g * u * g^{-1}) = \left(\Psi_F^{(i)}\right)^3(u)$, $i = 1,2,3, \dots, m$ and for all $u, g \in \acute{G}$. Now

$$\left(\varphi_F^{(i)}\right)^3(u * v) = \left(\varphi_F^{(i)}\right)^3(e * u * v)$$

$$= \left(\varphi_F^{(i)}\right)^3(v^{-1} * v * u * v)$$

$$= \left(\varphi_F^{(i)}\right)^3(v^{-1} * (v * u) * (v^{-1})^{-1})$$

$$= \left(\varphi_F^{(i)}\right)^3(v * u)$$

Hence, F is FmPFNSG on $(\acute{G},*)$.

Proposition:3.12 Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG on $(\acute{G},*)$ **.**Then $FmPFO(e) \leq FmPFO(a)$, for all $a \in \acute{G}$, where e is identity element of $(\acute{G},*)$.

Proof. Let $FmPFO(e) = n$, where $n \in N$.This implies $'\Omega(e)$ has n elements. Let

$'\Omega(e) = \{a_1, a_2, a_3, \dots, a_n\}$. This implies $\left(\varphi_F^{(i)}\right)^3(a_1) \geq \left(\varphi_F^{(i)}\right)^3(e)\ and\ \left(\Psi_F^{(i)}\right)^3(a_1) \leq \left(\Psi_F^{(i)}\right)^3(e)$, $\left(\varphi_F^{(i)}\right)^3(a_2) \geq \left(\varphi_F^{(i)}\right)^3(e)\ and\ \left(\Psi_F^{(i)}\right)^3(a_2) \leq \left(\Psi_F^{(i)}\right)^3(e)$

$,\dots, \left(\varphi_F^{(i)}\right)^3(a_n) \geq \left(\varphi_F^{(i)}\right)^3(e)\ and\ \left(\Psi_F^{(i)}\right)^3(a_n) \leq \left(\Psi_F^{(i)}\right)^3(e)$

Since, F is a FmPFSG on $(\acute{G},*)$ then $\left(\varphi_F^{(i)}\right)^3(e) \geq \left(\varphi_F^{(i)}\right)^3(a)\ and\ \left(\Psi_F^{(i)}\right)^3(e) \leq \left(\Psi_F^{(i)}\right)^3(a)$

for all $a \in \acute{G}$.This implies $\left(\varphi_F^{(i)}\right)^3(a_1) = \left(\varphi_F^{(i)}\right)^3(a_1) = \left(\varphi_F^{(i)}\right)^3(a_2) = \cdots = \left(\varphi_F^{(i)}\right)^3(a_n) = \left(\varphi_F^{(i)}\right)^3(e)\ and\ \left(\Psi_F^{(i)}\right)^3(a_1) = \left(\Psi_F^{(i)}\right)^3(a_2) = \cdots = \left(\Psi_F^{(i)}\right)^3(a_n) = \left(\Psi_F^{(i)}\right)^3(e)$.

Thus $'\Omega(e) \subseteq {}'\Omega(a)$, for all $a \in \acute{G}$.

Therefore $FmPFO(e) \leq FmPFO(a)$.

Definition:3.13 Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG on a group $(\acute{G},*)$ .The Fermatean m-polar fuzzy order of Fermatean m-polar Fuzzy subgroup F is defined by

$$FmPFO(F) = \vee\{FmPFO(a)/a \in \acute{G}\}.$$

Theorem:3.14 Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG on a group $(\acute{G},*)$. Then $O(a)\backslash(FmPFO(a))$.

Proof. Let $a \in \acute{G}$ and $O(a) = n$,where $n \in Z^+$. Then $a^n = e$. Consider $C = \langle a \rangle$ is a subgroup of $\acute{G}$.

Since F is a FmPFSG on a group $(\acute{G},*)$,

$$\left(\varphi_F^{(i)}\right)^3(a^2) = \left(\varphi_F^{(i)}\right)^3(a*a) \geq \left(\varphi_F^{(i)}\right)^3(a) \wedge \left(\varphi_F^{(i)}\right)^3(a) = \left(\varphi_F^{(i)}\right)^3(a) \text{ and}$$

$$\left(\Psi_F^{(i)}\right)^3(a^2) = \left(\Psi_F^{(i)}\right)^3(a*a) \leq \left(\Psi_F^{(i)}\right)^3(a) \vee \left(\Psi_F^{(i)}\right)^3(a) = \left(\Psi_F^{(i)}\right)^3(a), i = 1,2,\dots,m$$

By using induction, $\left(\varphi_F^{(i)}\right)^3(a^n) \geq \left(\varphi_F^{(i)}\right)^3(a)$ and $\left(\Psi_F^{(i)}\right)^3(a^n) \leq \left(\Psi_F^{(i)}\right)^3(a)$, for all $a \in \acute{G}$.

This implies, $a, a^2, a^3, \dots, a^n \in {}'\Omega(a)$. Consequently, $C \subseteq {}'\Omega(a)$. Therefore, C is a subgroup of $'\Omega(a)$.

Thus, by Lagrange's theorem, $O(C)\backslash O('\Omega(a))$. Therefore, $O(C)\backslash O('\Omega(a))$.

Theorem:3.15 Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG on a group $(\acute{G},*)$. Then $(FmPFO(a))\backslash O(\acute{G})$.

Proof. By the definition of $FmPFO$, $FmPFO(a) = O('\Omega(a))$.Since $'\Omega(a)$ subgroup of Ǵ,applying Lagrange's theorem we get $O('\Omega(a))\backslash O(Ǵ)$.This implies $FmPFO(a)\backslash O(Ǵ)$.

Theorem:3.16 Let $F = \left(\varphi_F^{(i)}(u), \Psi_F^{(i)}(u)\right), i = 1,2,3, \ldots, m$ be a FmPFSG on a group Ǵ . Then $FmPFO(a) = FmPFO(a^{-1})$.

Proof. Let $a \in$ Ǵ and F is a FmPFSG on a group on Ǵ.Then it satisfies

$\left(\varphi_F^{(i)}\right)^3(a^{-1}) \geq \left(\varphi_F^{(i)}\right)^3(a)$ and $\left(\Psi_F^{(i)}\right)^3(a^{-1}) \leq \left(\Psi_F^{(i)}\right)^3(a)$. Now $\left(\varphi_F^{(i)}\right)^3((a^{-1})^{-1}) \geq \left(\varphi_F^{(i)}\right)^3(a^{-1})$ and $\left(\Psi_F^{(i)}\right)^3((a^{-1})^{-1}) \leq \left(\Psi_F^{(i)}\right)^3(a^{-1})$.This implies $\left(\varphi_F^{(i)}\right)^3(a) \geq \left(\varphi_F^{(i)}\right)^3(a^{-1})$ and $\left(\Psi_F^{(i)}\right)^3(a) \leq \left(\Psi_F^{(i)}\right)^3(a^{-1})$.Thus $\left(\varphi_F^{(i)}\right)^3(a) = \left(\varphi_F^{(i)}\right)^3(a^{-1})$ and $\left(\Psi_F^{(i)}\right)^3(a) = \left(\Psi_F^{(i)}\right)^3(a^{-1})$. By definition $FmPFO(a) = O('\Omega(a))$ and

$$'\Omega(u) = \left\{v \in Ǵ \ / \ \left(\varphi_F^{(i)}\right)^3(v) \geq \left(\varphi_F^{(i)}\right)^3(u) \ and \ \left(\Psi_F^{(i)}\right)^3(v) \leq \left(\Psi_F^{(i)}\right)^3(u), i = 1,2,3, \ldots, m\right\}.$$

Then $'\Omega(a) = \left\{v \in Ǵ \ / \ \left(\varphi_F^{(i)}\right)^3(v) \geq \left(\varphi_F^{(i)}\right)^3(a) \ and \ \left(\Psi_F^{(i)}\right)^3(v) \leq \left(\Psi_F^{(i)}\right)^3(a), i = 1,2,3, \ldots, m\right\}$.This implies $'\Omega(a) = \left\{v \in Ǵ \ / \ \left(\varphi_F^{(i)}\right)^3(v) \geq \left(\varphi_F^{(i)}\right)^3(a^{-1}) \ and \ \left(\Psi_F^{(i)}\right)^3(v) \leq \left(\Psi_F^{(i)}\right)^3(a^{-1}), i = 1,2,3, \ldots, m\right\}$.This implies $'\Omega(a) = {}'\Omega(a^{-1})$.That is $O('\Omega(a)) = O('\Omega(a^{-1}))$.Thus $FmPFO(a) = FmPFO(a^{-1})$.

Proposition:3.17 Let $F_1 = \left(\varphi_{F_1}^{(i)}(u), \Psi_{F_1}^{(i)}(u)\right)$ and $F_2 = \left(\varphi_{F_2}^{(i)}(u), \Psi_{F_2}^{(i)}(u)\right)$, $i = 1,2,3, \ldots, m$ be a FmPNFSG on a group $(Ǵ,*)$. Then $F_1 \cap F_2$ is FmPNFSG.

Proof. Let $F = F_1 \cap F_2$.Since $F_1 \ and \ F_2$ are FmPNFSG, then $aF_1 = F_1a$ and $aF_2 = F_2a$.

This implies $\left(\varphi_{aF_1}^{(i)}(u)\right)^3 = \left(\varphi_{F_1a}^{(i)}(u)\right)^3, \left(\Psi_{aF_1}^{(i)}(u)\right)^3 = \left(\Psi_{F_1a}^{(i)}(u)\right)^3$ and $\left(\varphi_{aF_2}^{(i)}(u)\right)^3 = \left(\varphi_{F_2a}^{(i)}(u)\right)^3, \left(\Psi_{aF_2}^{(i)}(u)\right)^3 = \left(\Psi_{F_2a}^{(i)}(u)\right)^3$ $i = 1,2,3, \ldots, m$ and for all $a, u \in Ǵ$, where $\left(\varphi_{aF}^{(i)}(u)\right)^3 = \left(\varphi_F^{(i)}(a^{-1}u)\right)^3, \left(\varphi_{Fa}^{(i)}(u)\right)^3 = \left(\varphi_F^{(i)}(ua^{-1})\right)^3$ and

$$\left(\Psi_{aF}^{(i)}(u)\right)^3 = \left(\Psi_F^{(i)}(a^{-1}u)\right)^3, \left(\Psi_{Fa}^{(i)}(u)\right)^3 = \left(\Psi_F^{(i)}(ua^{-1})\right)^3$$

Now, we have to prove that $\left(\varphi_{aF}^{(i)}(u)\right)^3 = \left(\varphi_{Fa}^{(i)}(u)\right)^3$ and $\left(\Psi_{aF}^{(i)}(u)\right)^3 = \left(\Psi_{Fa}^{(i)}(u)\right)^3$.

Consider, $\left(\varphi_{aF}^{(i)}(u)\right)^3 = \left(\varphi_{a(F_1 \cap F_2)}^{(i)}(u)\right)^3$

$$= \left(\varphi_{F_1 \cap F_2}^{(i)}(a^{-1}u)\right)^3 = min\left\{\left(\varphi_{F_1}^{(i)}(a^{-1}u)\right)^3, \left(\varphi_{F_2}^{(i)}(a^{-1}u)\right)^3\right\}$$

$$= min\left\{\left(\varphi_{F_1}^{(i)}(ua^{-1})\right)^3, \left(\varphi_{F_2}^{(i)}(ua^{-1})\right)^3\right\}$$

$$= \left(\varphi^{(i)}_{(F_1\cap F_2)a}(u)\right)^3$$

$$= \left(\varphi^{(i)}_{Fa}(u)\right)^3 \qquad \text{and}$$

$$\left(\Psi^{(i)}_{aF}(u)\right)^3 = \left(\Psi^{(i)}_{a(F_1\cap F_2)}(u)\right)^3$$

$$= \left(\Psi^{(i)}_{F_1\cap F_2}(a^{-1}u)\right)^3$$

$$= max\left\{\left(\Psi^{(i)}_{F_1}(a^{-1}u)\right)^3, \left(\Psi^{(i)}_{F_2}(a^{-1}u)\right)^3\right\}$$

$$= max\left\{\left(\Psi^{(i)}_{F_1}(ua^{-1})\right)^3, \left(\Psi^{(i)}_{F_2}(ua^{-1})\right)^3\right\}$$

$$= \left(\Psi^{(i)}_{(F_1\cap F_2)a}(u)\right)^3$$

$$= \left(\Psi^{(i)}_{Fa}(u)\right)^3$$

Hence $F_1 \cap F_2$ is FmPNFSG.

Proposition:3.18 Let Ǵ be a group of order n and $F = \left(\varphi^{(i)}_F(u), \Psi^{(i)}_F(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG on a group Ǵ.Then there exists an element $x \in$ Ǵ such that $\left(\varphi^{(i)}_F\right)^3(x) \le \left(\varphi^{(i)}_F\right)^3(y)$ and $\left(\Psi^{(i)}_F\right)^3(x) \ge \left(\Psi^{(i)}_F\right)^3(y)$, for all $y \in$ Ǵ.

Proof. Let Ǵ $= \{a_1, a_2, \dots, a_n\}$ be a group and $F = \left\{\langle a_j, \left(\varphi^{(i)}_F(a_j), \Psi^{(i)}_F(a_j)\right)\rangle\right\}, i = 1,2,3,\dots,m\,; j = 1,2,\dots,n$ be FmPFSG on a group Ǵ.

Let $A^i = \left\{\varphi^{(i)}_F(a_1), \varphi^{(i)}_F(a_2), \dots, \varphi^{(i)}_F(a_n)\right\}, k = 1,2,\dots,m$.Since A^i has finite number of elements and $\varphi^{(i)}_F(a_j) \in [0,1]$,then A^i has at least one minimum value.

Therefore, there exists an element $x \in$ Ǵ such that $\left(\varphi^{(i)}_F\right)^3(x) \le \left(\varphi^{(i)}_F\right)^3(y)$, for all $y \in$ Ǵ.

Similarly, we can prove there exists an element $x \in$ Ǵ such that $\left(\Psi^{(i)}_F\right)^3(x) \ge \left(\Psi^{(i)}_F\right)^3(y)$, for all $y \in$ Ǵ.

Theorem:3.19 Let $o($Ǵ$) = n$ and $F = \left(\varphi^{(i)}_F(u), \Psi^{(i)}_F(u)\right), i = 1,2,3,\dots,m$ be a FmPFSG on a group Ǵ . Then $FmPFO(F) = O($Ǵ$)$.

Proof. Let Ǵ $= \{a_1, a_2, \dots, a_n\}$ be a group and $o($Ǵ$) = n$.

By the definition of $FmPFO$, $FmPFO($Ǵ$) = \vee\left\{O\left(\Omega(a_j)\right)/a_j \in \text{Ǵ}\right\}$.From the Preposition, there exists an element $a_k \in$ Ǵ such that $\left(\varphi^{(i)}_F\right)^3(a_k) \le \left(\varphi^{(i)}_F\right)^3(a_j)$ and $\left(\Psi^{(i)}_F\right)^3(a_k) \ge \left(\Psi^{(i)}_F\right)^3(a_j), j = 1,2,\dots,n$.

$$\text{Now } \Omega(a_k) = \left\{ a_j \in \acute{G} / \left(\varphi_F^{(i)}\right)^3 (a_j) \geq \left(\varphi_F^{(i)}\right)^3 (a_k) \text{ and } \left(\varphi_F^{(i)}\right)^3 (a_j) \geq \left(\varphi_F^{(i)}\right)^3 (a_k) \right\}$$

$$\text{This implies } O(\Omega(a_k)) = n.$$

$$\text{Therefore, } FmPFO(F) = n = O(\acute{G}).$$

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