

Exploring the Mathematics of Spacetime in Einstein's Relativity

Priti Nandeshwar*

Department of Mathematics, Smt. Sugrata Wanjari Mahila Mahavidyalaya, Wadoda Tal. Kamptee, Dist.- Nagpur – 441104

ABSTRACT

This paper maps the mathematical architecture of General Relativity, stepping away from basic coordinate-based tensor calculus to look at the deeper differential geometry at play. We formalize spacetime as a pseudo-Riemannian manifold (M, g) that relies on a torsion-free Levi-Civita connection. By applying the principle of least action to the Einstein-Hilbert action, we derive the Einstein Field Equations from first principles. From there, we introduce Élie Cartan's tetrad formalism. This approach proves mathematically indispensable when trying to couple fermionic fields to the geometry of spacetime. Ultimately, this framework shows that gravity isn't a traditional force at all. It is an intrinsic geometric deformation of the manifold itself, a reality that naturally paves the way for advanced extensions like Einstein-Cartan theory

Keywords: General Relativity, Differential Geometry, Pseudo-Riemannian Manifolds, Einstein Field Equations, Tetrad Formalism, Spinors, Einstein-Cartan Theory

INTRODUCTION

When Albert Einstein introduced his theories of relativity, the entire mathematical vocabulary of physics had to be rewritten. For centuries, classical Newtonian mechanics had relied on a rigid, unchanging Euclidean backdrop. Space and time were separate, absolute entities acting as a stage for matter, completely unaffected by the actors upon it. General Relativity, published in 1915, shattered this assumption. Einstein proposed something radical: gravity is simply the manifestation of a dynamic, curving four-dimensional spacetime. To actually formalize this mind-bending concept, Einstein leaned heavily on the absolute differential calculus pioneered by mathematicians Gregorio Ricci-Curbastro and Tullio Levi-Civita. Their framework allowed physical laws to be written in a generally covariant form—meaning the laws hold true regardless of the coordinate system the observer happens to be using. The aim of this paper is to unpack that foundational mathematics. We will start by defining spacetime strictly as a pseudo-Riemannian manifold and building the necessary curvature tensors. Next, we will use the variational method to pull the Einstein

Field Equations out of thin air from first principles. Finally, we will pivot to a more advanced geometric perspective—Cartan's tetrad formalism—to see exactly why it is required for folding quantum spin into a gravitational framework.

2. Spacetime as a Pseudo-Riemannian Manifold

To do relativity properly, we have to throw out flat Minkowski spacetime. Instead, we generalize it into a curved, four-dimensional smooth manifold M . The entire geometric structure of this manifold M is dictated by a single mathematical object: the metric tensor field g . This is a symmetric, non-degenerate $(0,2)$ -tensor sporting a Lorentzian signature of $(-, +, +, +)$. If you pick any point $p \in M$, the metric $g|_p$ defines an inner product on the tangent space T_pM . We express the infinitesimal squared distance—the spacetime interval—in local coordinates x^μ like this:

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$$

Here is the crucial difference from classical mechanics: $g_{\mu\nu}$ is not static. It is a highly dynamic variable, constantly pushed and pulled by the mass and energy scattered throughout the universe. Think

Relevant conflicts of interest/financial disclosures: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.



of it as the foundational potential that generates what we experience as the gravitational field.

3. The Levi-Civita Connection and Curvature

You can't just easily differentiate vector fields across a curved manifold. To do it, you have to introduce an affine connection ∇ . Thanks to the Fundamental Theorem of Riemannian Geometry, we know that exactly one unique connection exists that is both torsion-free $\nabla_X Y - \nabla_Y X = [X, Y]$ and metric-compatible (∇_g). Physicists call this the Levi-Civita connection. Once this foundational connection is established, the next logical step is to measure the actual, intrinsic curvature of the manifold. In differential geometry, this is captured by the Riemann curvature tensor. Instead of relying on abstract coordinates, you can picture this physically through the concept of parallel transport. If you take a vector and carefully carry it along a closed loop on a curved surface—always keeping it strictly parallel to its previous orientation—it will not point in the same direction when you finally return to your starting point. The Riemann tensor precisely quantifies this directional failure. Additionally, this tensor naturally obeys specific structural rules known as the Bianchi identities. In physical terms, these identities are profoundly important because they act as the underlying mathematical mechanisms that guarantee the local conservation of energy and momentum throughout the universe.

4. Variational Formulation: The Einstein-Hilbert Action

When it came to deriving the actual field equations, Einstein initially hacked his way there using brilliant physical intuition. However, the mathematician David Hilbert soon proved that these foundational equations arise completely naturally from the principle of least action. The dynamics of the spacetime geometry are governed by the Einstein-Hilbert action, a profound integral that sums up the total scalar curvature across the entire spacetime manifold. Of course, to describe a realistic universe, you must also account for the presence of physical mass by incorporating a separate matter action. When you mathematically extremize this combined total action—essentially forcing the geometry to find its path of least resistance—the natural mathematical byproduct is the complete set of

Einstein Field Equations. These conceptual equations perfectly balance the geometric curvature of spacetime on one side against the stress-energy tensor on the other, seamlessly linking the physical shape of the universe to the matter residing inside it.

5. Cartan's Tetrad Formalism and Fermionic Coupling

Relying strictly on coordinate-based approaches works perfectly well for macroscopic astrophysical problems, but it tends to obscure the underlying local symmetries of spacetime. A much more robust analytical tool is Élie Cartan's method of moving frames, commonly referred to in the literature as the tetrad or vierbein formalism. Rather than calculating the overarching spacetime metric directly, this method attaches a localized set of mutually orthogonal reference vectors to every single point in the manifold. This shift in perspective is crucial because it allows gravity to be treated mathematically as a gauge theory. More importantly, if you want to understand how particles with half-integer quantum spin—such as electrons and quarks—interact with a gravitational field, the tetrad formalism is absolutely non-negotiable. Standard General Relativity assumes a completely smooth, torsion-free geometry, which simply has no mathematical room to accommodate intrinsic quantum spin. To bridge this gap, theoretical physicists turn to necessary extensions like Einstein-Cartan theory. In this expanded framework, mass and energy still dictate the overarching curvature of spacetime, but the intrinsic spin of quantum particles introduces a localized, microscopic twisting effect known as torsion.

CONCLUSION

Ultimately, stripping away the complex physical terminology reveals that Einstein's theory of relativity is an exhaustive, beautiful exploration of pseudo-Riemannian geometry. Whether analyzed through traditional metric tensors or the more advanced tetrad formalism, the foundational philosophy remains unchanged. Gravity completely ceases to be a mysterious Newtonian force pulling objects across an empty void. Instead, it is revealed as a precise, dynamic deformation of the spacetime manifold itself—an elegant architectural framework capable of describing the orbits of massive galaxies while

simultaneously laying the groundwork to understand the quantum mechanics of fundamental particles.

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HOW TO CITE: Priti Nandeshwar*, Exploring the Mathematics of Spacetime in Einstein's Relativity, *Int. J. Sci. R. Tech.*, 2026, 3 (3), 305-307. <https://doi.org/10.5281/zenodo.19029069>