Int. J. Sci. R. Tech., 2026 3(7)
A Multidisciplinary peer-reviewed Journal
www.ijsrtjournal.com [ISSN: 2394-7063]

Original Article

Some Significant Novel Identities Of Fractional Laplace Transform

Shilpi Agrawal*, Rajshree Mishra

Govt. Kamalaraja Girls PG Autonomous College Gwalior pincode 474001

ABSTRACT

Fractional calculus has emerged as an important research area due to its extensive applications in science, engineering and mathematical physics. Among various integral transforms, the fractional Laplace transform has attracted considerable attention because of its capability to solve fractional differential equations and other complex mathematical models more effectively than the classical Laplace transform. Although Jummarie introduced the fractional Laplace transform and established its convolution theorem, several fundamental identities associated with this transform remain unexplored. This paper aims to enrich the theoretical framework of the fractional Laplace transform by deriving two important results: the Product theorem and Parseval's identity. The proposed derivations are developed using the properties of the Mittag–Leffler function and the inverse fractional Laplace transform within the framework of fractional calculus. These identities extend the analytical capabilities of the fractional Laplace transform and provide a stronger mathematical foundation for its application in signal processing, control theory, mathematical physics, and engineering problems involving fractional-order systems. The obtained results are expected to facilitate further theoretical developments and computational applications of the fractional Laplace transform

Keywords: Fractional Laplace Transform, Fractional Calculus, Mittag–Leffler Function, Product Theorem, Parseval's Identity, Fractional Convolution.

INTRODUCTION

Fractionalization of transform has been an attention grabbing topic for the scientists since last four decades, because of its better performance as compare to classical transform. Infact fractionalization of a transform makes it much competent and powerful tool in various domains of science and engineering. In this context one of the most effective methods for handling several physical issues in mathematical physics, applied mathematics, and engineering is the Laplace transform. Because of its amazing applications in the field of dynamic and delayed differential equations, it attracted significant interest from researchers [1-10]. In 2009 Jummarie introduced the fractional Laplace transform and provided a convolution theorem of fractional Laplace transform along with some applications [11].

Even after the rigrous work by Jummarie in 2009, only a few investigators paid attention on fractionalization of Laplace transform. Some researchers mainly K.K. Sharma, Huang, Tan, Lepag, Schiff and Rudolf presented applications on fractional Laplace transforms [12, 13-17]. Recently Teekam et. al. applied the fractional Laplace transform given by Jummarie to solve fractional wave and heat equtions [18]. Although Jummarie presented consolidated definition of fractional Laplace transform but over the period of 15 years there is still a scope of developing its analytical theory produce.

Purpose of present paper is to derive Product theorem and Parseval's identity of fractional Laplace transform given by Jummarie using Mittage-Leffler function in enviornment of fractional calculus.

Jummarie introduced a novel definition of fractional Laplace transform and produced some properties of this newly introduced transform [11]. But many identities of the transform are still missing. In the present paper we have attempted to derive product theorem and parseval's identity of this novel Laplace transform of fractional order.

Relevant conflicts of interest/financial disclosures: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

2. PRELIMINARIES

[I] Laplace Transform

The Laplace transform of the function $f(t)$ is denoted by F(s)and is defined as [2]

$F(s)= \int_0^{+\infty} e^{-st} f(t)\, dt, \quad s > 0$ (1.1)

[II] Inverse Laplace Transform [2]

If F(s) is the Laplace transform of a function $f(t)$, *i.e.* $L\{f(t)\}=F(s)$

Then $f(t)$ is called the inverse Laplace transform of the function F(s) and is written as [2]

$$f(t)=L^{-1}\{F(s)\}$$

L^{-1} *is called the inverse Laplace transform operator.*

It is inverse complex formula

If $f(t)$ has a continuous derivative and is exponential order and if

$L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by

$f(t)= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma-i\infty} e^{st} F(s)ds, \quad t \geq 0, \ \gamma >$ real part of all poles of F (1.2)

and $f(t)= 0$ for $t < 0$.

This result is called the complex inversion integral or formula. It is also known as Bromwich's integral formula.

[III] Fractional Laplace Transform

Let $f(x)$ denote a function which vanishes for negative values of x, *then* its Laplace's transform $L_\alpha\{f(x)\}$ of order α (or its α-th fractional Laplace's transform) is defined by the following expression [11].

$L_\alpha\{f(x)\} = F_\alpha(s) = \int_0^{\infty} E_\alpha(-s^\alpha x^\alpha)\, f(x)(dx)^\alpha$ (1.3)

Provided the integral converges. It can also be expressed as

$= lim\ M \uparrow \infty \quad \int_0^{M} E_\alpha(-s^\alpha x^\alpha)\, f(x)(dx)^\alpha$ (1.4)

where $s \in C$, and $E_{\alpha(u)}$is the Mittage-Leffler function , for $\alpha > 0$ is given by [11]

$$E_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(\alpha k+1)}, \quad (1.5)$$

where $\alpha \in C$, $u \in C$ and $Re(\alpha) > 0$. The Mittag-Leffler function $E_\alpha(u)$ is an entire function of type 1, which reduces to exponential function $e^u = E_1(u)$ *for* $\alpha=1$.

We will use following identity of Mittage-Leffler function [11]

$$E_\alpha(\lambda(x+y)^\alpha = E_\alpha(\lambda x^\alpha)E_\alpha(\lambda y^\alpha).$$

[IV] Inverse Fractional Laplace Transform

Given the Laplace's transform (1.3) that we recall here for convenience [11]:

$F_\alpha(s) = \int_0^{+\infty} E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha, \quad 0 < \alpha < 1,$ (1.6)

one has the inverse formula

$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{+i\infty} E_\alpha(s^\alpha x^\alpha) F_\alpha(s)(ds)^\alpha$ (1.7)

3. SOME NOVEL IDENTITIES OF FRLT

In this section we have attempted to derive some identities of FRLT.

Following definition of fractional convolution was given by Jummarie [11]

Fractional Convolution

Convolution of order α of the two functions $f(x)$ and $g(x)$ is defined by [11]

$(f(x) * g(x))_\alpha = \int_0^{x} f(x-u)g(u)(du)^\alpha.$ (1.8)

He presented following convolution theorem using the definition of fractional convolution (1.7) for fractional Laplace transform [11].

Convolution Theorem of Fractional Laplace Transform:

"Fractional Laplace transform of convolution of two functions is equal to the product of that transforms."

i.e. $L_\alpha\{(f(x) * g(x))_\alpha\} = L_\alpha\{f(x)\}L_\alpha\{g(x)\}$.

This result is the most extensively used phenomena of convolution property of transforms. Jummarie established the convolution theorem but many identities like product theorem and Parseval's identity are still lacking in the theory of fractional Laplace transforms.

This article aims to enrich the theory of fractional Laplace transform by producing two significant theorems i.e. Product theorem and Parseval's identity.

Theorem [1] Product Theorem of Fractional Laplace Transform

If $L_\alpha\{f(x) * g(x)\}_\alpha = \frac{1}{(2\pi)^\alpha}(F_\alpha * g_\alpha)_\alpha(s)$, (1.9)

i.e. the transform of the product is the convolution of the transforms.

Proof

Let $L_\alpha\{f(x)\} = F_\alpha(s)$,

$$L_\alpha\{g(x)\} = G_\alpha(s)$$

$$L_\alpha\{f(x).g(x)\}_\alpha = \int_0^\infty E_\alpha(-s^\alpha x^\alpha)\ f(x)g(x)(dx)^\alpha. \qquad \text{using (1.3)}$$

Inverse transform of $g(x)$

$$g(x) = \frac{1}{(2\pi)^\alpha}\int_0^\infty E_\alpha(\sigma^\alpha x^\alpha)\, G_\alpha(\sigma)(d\sigma)^\alpha$$

$$L_\alpha\{f(x) * g(x)\}_\alpha = \frac{1}{(2\pi)^\alpha}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)\ f(x)\left[\int_0^\infty E_\alpha(\sigma^\alpha x^\alpha)G_\alpha(\sigma)(d\sigma)^\alpha\right](dx)^\alpha.$$

Interchange the integrals

$$\frac{1}{(2\pi)^\alpha}\int_0^\infty G_\alpha(\sigma)\left[\int_0^\infty E_\alpha(-s^\alpha x^\alpha)E_\alpha(\sigma^\alpha x^\alpha)f(x)(dx)^\alpha\right](d\sigma)^\alpha$$

Use the Mittage-Leffler property [11]

$$E_\alpha(a^\alpha x^\alpha)E_\alpha(b^\alpha x^\alpha) = E_\alpha((a+b)^\alpha x^\alpha)$$

We get

$$E_\alpha(-s^\alpha x^\alpha)E_\alpha(\sigma^\alpha x^\alpha) = E_\alpha[-(s-\sigma)^\alpha x^\alpha]$$

$$= \frac{1}{(2\pi)^\alpha}\int_0^\infty G_\alpha(\sigma)\left[\int_0^\infty E_\alpha[-(s-\sigma)^\alpha x^\alpha]f(x)(dx)^\alpha\right](d\sigma)^\alpha$$

$$= \frac{1}{(2\pi)^\alpha}\int_0^\infty F_\alpha(s-\sigma)\, G_\alpha(\sigma)\,(d\sigma)^\alpha$$

By definition of fractional convolution

$$(F_\alpha * g_\alpha)_\alpha(s) = \int_0^\infty F_\alpha(s-\sigma)\, G_\alpha(\sigma)\,(d\sigma)^\alpha$$

Hence,

$$L_\alpha\{f(x).g(x)\}_\alpha = \frac{1}{(2\pi)^\alpha}(F_\alpha * g_\alpha)_\alpha(s)$$

Theorem [2] Parseval's Identity of Fractional Laplace Transform

If $L_\alpha\{f(x)\} = F_\alpha(s)$,

and $L_\alpha\{g(x)\} = G_\alpha(s)$,

If $f(x)$ and $g(x)$ are piece wise continuous and their fractional Laplace transform form exist, then the identity.

$$\int_0^\infty f(x)g(x)(dx)^\alpha = \frac{1}{(2\pi)^\alpha}\int_0^\infty F_\alpha(s)\, G_\alpha(-s)(d\, s)^\alpha$$ (1.10)

Proof

Let $L_\alpha\{f(x)\} = F_\alpha(s), L_\alpha\{g(x)\} = G_\alpha(s)$,

Using the inverse fractional Laplace transform.

$$g(x)=\frac{1}{(2\pi)^\alpha}\int_0^\infty E_\alpha(s^\alpha x^\alpha)\, G_\alpha(s)(d\, s)^\alpha$$

Multiplying both sides by $f(x)$

$$f(x)g(x) = \frac{1}{(2\pi)^\alpha} f(x)\int_0^\infty E_\alpha(s^\alpha x^\alpha)\, G_\alpha(s)(d\, s)^\alpha$$

$$\int_0^\infty f(x)\, g(x)(dx)^\alpha = \frac{1}{(2\pi)^\alpha}\int_0^\infty \left[\int_0^\infty f(x)E_\alpha(s^\alpha x^\alpha)\,(dx)^\alpha\right] G_\alpha(s)(d\, s)^\alpha$$

Changing the order of integration and using the transform definition.

$$\int_0^\infty f(x)\, E_\alpha(s^\alpha x^\alpha)\,(dx)^\alpha = F_\alpha(-s)$$

Therefore,

$$\int_0^\infty f(x)\, g(x)(dx)^\alpha = \frac{1}{(2\pi)^\alpha}\int_0^\infty F_\alpha\,(-s)G_\alpha(s)(ds)^\alpha$$

$$\int_0^\infty f(x)\, g(x)(dx)^\alpha = \frac{1}{(2\pi)^\alpha}\int_0^\infty F_\alpha\,(s)g_\alpha(-s)(ds)^\alpha$$

Hence,

$$\int_0^\infty f(x)\, g(x)(dx)^\alpha = \frac{1}{(2\pi)^\alpha}\int_0^\infty F_\alpha\,(s)g_\alpha(-s)(ds)^\alpha$$

CONCLUSION

Fractional Laplace transform is also an area of interest due to its applicability and modifiability in diverse disciplines of engineering and sciences but theoretical development of fractional Laplace transform still needs attention of investigators. Parseval identity of transforms proved to be significant phenomena in signal processing area. Similarly, Product theorem is also very important for specific applications. We have attempted these two significant theorems for fractional Laplace transform. Digital computation of our results can engendre variety of applications in multiple sectors of science and engineering.

REFERENCES

1. Sneddon, I.N. The use of Integral transforms. New York, McGraw-Hill. 1972.
2. Debnath, L. and Bhatta, D.Integral transform and their applications. 2nd Ed. Chapman & Hall/CRC.2007;https://doi.org/10.1201/9781420010916.
3. Prabhakar R. Deshmukh, Alka S. Gudadhe," Convolution structure for two version of fractional Laplace transform,"Journal of Science and Arts year 11, no. 2(15),pp.143-150, (2011).

4. Likexue Peng Jigen,"Laplace transform and fractional differential equations," Applied Mathematics Letters, Volume 24, Issue 12,Pages 2019-2023, (2011).
5. A.M.O. Anwar, F. Jarad, D. Baleanu, F. Ayaz," Fractional caputo heat equation with in the double Laplace transform," Rom. Journ. Phys., Vol. 58, Nos. 1–2, P. 15–22, Bucharest, (2013).
6. P. V. Murlidhar, Y. Srinivasaro & M. S. R. Naidu, Convolution theorem for fractional Laplace transform,"International Journal of Electronics, Communication & Instrumentation Engineering Research and Development (IJECIERD) ISSN 2249-684X Vol. 3, Issue 4,(2013).
7. Nelson Ricardo Ojeda, Luis E Guillermo Romero, Fractional Laplace transform and fractional Calculus, International Mathematical Forum, Jan. 2017.
8. Fahd Jarad, and Thabet Abdeljawad," Generalized fractional derivatives and Laplace transform,"Americal Institute of Mathematical Science, March 2020,13(3),709-722.
9. Chenkuan Li, On the generalized fractional Laplacian, fractional calculus and Applied Analysis 24(6), 1797-1830, 2021.
10. Asa Omer, Laplace transform solutions for heat and mass transfer problem using Caputo-Fabrizio fractional derivative, European Journal of Pure and Applied Mathematics 19(1), 7245-7245, 2026.
11. Jumarie, G. Laplace's Transform of Fractional Order Via the Mittage-Leffler Function and Modified Riemann-Liouville Derivative. AppliedMathematicsLetters. 2008; 22: 1659-1664,https://doi.org/10.1016/j.aml.2009.05.011.
12. Sharma, K. K. Fractional Laplace transform. Signal, image and video processing. 2010, 4(3), 377-379.
13. Huang, C., Yang, Z., Yi, T. & Zou, X. (2014). On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities. Journal of Differential Equations. 256(7), 2101-2114. https://doi.org/10.1016/j.jde.2013.12.015.
14. Tan, Y., Huang, C., Sun, B., & Wang, T. (2018). Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition. Journal of Mathematical Analysis and Applications. 458(2), 1115-1130. https://doi.org/10.1016/j.jmaa.2017.09.045.
15. Le Page, W. R. (1980). Complex variables and the Laplace transform for engineers. Courier Corporation.
16. Schiff, J. L. (1999). The Laplace transform, theory and applications. Springer Science and Business Media.
17. Rudolf A. Treumann and Wolfgang Baumjohann," fractional Laplace transforms—a perspective,"Perspective Article, Front. Phys., (2014).
18. Mahor, Teekam Chand. A Study of Fractional Integral Transforms and Their Applications. Ph.D. Thesis, Jiwaji University, Gwalior, 2022. Guide: Rajshree Mishra.

HOW TO CITE: Shilpi Agrawal*, Rajshree Mishra, Some Significant Novel Identities Of Fractional Laplace Transform, Int. J. Sci. R. Tech., 2026, 3 (7), 869-873. https://doi.org/10.5281/zenodo.21534487